Sussan

colored Jones polynomial

$U_q(sl_2)$ / $\mathbb{C}(q)$-alg with generators $E, F, K^{\pm}$

relations $KE = q^2 EK$

$KF = q^{-2} FK$

$KK^{-1} = id = K^{-1}K$

$EF - FE = \dfrac{K - K^{-1}}{q - q^{-1}}$

V_n : $\mathbb{C}(q)$-vector space

with base $\{v_0, \dots, v_n\}$

This has a structure of a $U_q(sl_2)$-repr.

$V_{d_1} \otimes \cdots \otimes V_{d_r}$ is also a $U_q(sl_2)$-repr.

$\cap : V_1^{\otimes 2} \to \mathbb{C}(q)$

$\cap(v_0 \otimes v_1) = 1$

$\cap(v_1 \otimes v_0) = -q^{-1}$ $\qquad \cap(v_1 \otimes v_1) = \cap(v_0 \otimes v_0) = 0$

$\cap_{j,n} : V_1^{\otimes n} \to V_1^{\otimes(n-2)}$

$\quad = id^{\otimes(j-1)} \otimes \cap \otimes id^{\otimes(\ )}$

$\cup : \mathbb{C}(q) \to V_1^{\otimes 2}$ $\qquad \cup(1) = v_1 \otimes v_0 - q v_0 \otimes v_1$

$\cup_{j,n} : V_1^{\otimes n} \to V_1^{\otimes(n+2)}$

$\times : V_1^{\otimes 2} \to V_1^{\otimes 2}$ $\qquad -q^2 id - q\, {}^{\cup}_{\cap}$

$\times_{j,n} : V_1^{\otimes n} \to V_1^{\otimes n}$

$\times = -q^{-2} id - q^{-1}\, {}^{\cup}_{\cap}$ $\qquad \times_{j,n}$

To each (r,s) tangle diagram D we assign an intertwiner

$\varphi(D) : V_1^{\otimes r} \to V_1^{\otimes s}$ by $||\cap|| \to \cap_{j,n}$

etc

Let D be an oriented tangle diagram.

$\gamma(D) = \#$ of crossings of type  or 

$- \#$ of crossings of type  or 

Th T : oriented (r,s) tangle

D_1, D_2 two planar projections

$$\Longrightarrow \quad q^{3\gamma(D_1)} \varphi(D_1) = q^{3\gamma(D_2)} \varphi(D_2) : V_1^{\otimes r} \to V_1^{\otimes s}$$

In the special case of a $(0,0)$-tangle we get the Jones polynomial

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$\iota_n : V_n \longrightarrow V_1^{\otimes n}$

$$\iota_n(v_k) = \sum_{\substack{\vec{d} \\ |\vec{d}|=k}} c_1(\vec{d})\, v_{d_1} \otimes \cdots \otimes v_{d_n}$$

$\vec{d} = (d_1, \dots, d_n) \qquad d_i = 0 \text{ or } 1$

$c_1 : \{0,1\}^n \to \mathbb{C}(q)$

$\pi_n : V_1^{\otimes n} \longrightarrow V_n$

$$\pi_n(v_{d_1} \otimes \cdots \otimes v_{d}) = c_2(\vec{d})\, v_{|\vec{d}|}$$

$c_2(\vec{d}) \in \mathbb{C}(q)$

Now consider an oriented (r,s) tangle with each of the strands colored by various finite dim. irr reps

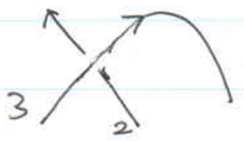
This induces a coloring on the end points

(r,s)-tangle $\rightsquigarrow$ $(d_1, \dots, d_r), (e_1, \dots, e_s)$-tangles

Want a map $\quad V_{d_1} \otimes \cdots \otimes V_{d_r} \longrightarrow V_{e_1} \otimes \cdots \otimes V_{e_s}$

to each colored oriented diagram D

we associate its cabled diagram $\mathrm{cab}(D)$



D

$\mathrm{cab}(D)$

To an $((d_1,\dots,d_r),(e_1,\dots,e_s))$ tangle diagram D

We associate a map $\mathcal{E}_{col}(D) : V_{d_1}\otimes\cdots\otimes V_{d_r} \to V_{e_1}\otimes\cdots\otimes V_{e_s}$

$$= \pi_{e_1}\otimes\cdots\otimes\pi_{e_s}\,\mathcal{E}(\mathrm{cab}\,D)\circ(i_{d_1}\otimes\cdots\otimes i_{d_r})$$

Th T be an oriented, framed $((d_1,\dots,d_r),(e_1,\dots,e_s))$-tangle

D_1, D_2 two of its diagrams

$$\Rightarrow \quad q^{3\delta(\mathrm{cab}(D_1))}\mathcal{E}_{col}(D_1) = q^{3\delta(\mathrm{cab}\,D_2)}\mathcal{E}_{col}(D_2)$$

$$: V_{d_1}\otimes\cdots\otimes V_{d_r} \to V_{e_1}\otimes\cdots\otimes V_{e_s}$$

In the case of $(0,0)$-tangle, this is the colored Jones pol.

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categorification of Jones polynomial

$\mathcal{O}_i(\mathfrak{gl}_n) = \mathcal{O}_\lambda(\mathfrak{gl}_n)$ $\qquad \lambda = e_1+\cdots+e_i-\rho$

this can be viewed as a category of graded modules

Th (BFKS)

1) $\mathbb{C}(q)\otimes_{\mathbb{Z}[q,q^{-1}]}\left[\bigoplus_{i=0}^{n}\mathcal{O}_i(\mathfrak{gl}_n)\right] \cong V_1^{\otimes n}$

2) projective functors $\mathcal{E},\mathcal{F},\mathcal{K},\mathcal{K}^{-1}$ which satisfy functorial $U_q(\mathfrak{sl}_2)$-relation

$\mathcal{O}_i^j(\mathfrak{gl}_n)$ = full subcategory of $\mathcal{O}_i(\mathfrak{gl}_n)$
of modules loc. finite with respect to j- 

$\varepsilon_j : \mathcal{O}_i^j(\mathfrak{gl}_n) \to \mathcal{O}_i(\mathfrak{gl}_n)$ inclusion functor
$Z_j : \mathcal{O}_i(\mathfrak{gl}_n) \to \mathcal{O}_i^j(\mathfrak{gl}_n)$

It takes M to its maximal locally finite quot. w.r.t. 

Z_j is right exact.
We consider the left adjoint functor

$$\mathcal{L}Z_j : D^b(\mathcal{O}_i(\mathfrak{gl}_n)) \to D^b(\mathcal{O}_i^j(\mathfrak{gl}_n))$$

Th (BFK)
$\exists$ equiv. of categories F_1, F_2 s.t.

1) $\left[\widetilde{\cap}_{j,n} \overset{\text{def.}}{=} F_1 \circ \mathcal{L}Z_j\right] = \cap_{j,n}$

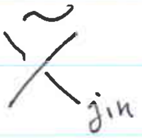
$$\widetilde{\cap}_{j,n} : D^b(\oplus_i \mathcal{O}_i(\mathfrak{gl}_n)) \to D^b(\oplus_i \mathcal{O}_i(\mathfrak{gl}_{n-2}))$$

2) $\left[\widetilde{\cup}_{j,n} \overset{\text{def.}}{=} \varepsilon_j[-1] \circ F_2\right] = \cup_{j,n}$

$$\widetilde{\cup}_{j,n} : D^b(\oplus_i \mathcal{O}_i(\mathfrak{gl}_n)) \to D^b(\oplus_i \mathcal{O}_i(\mathfrak{gl}_{n+2}))$$

There exists natural transformations

$$\begin{array}{ccccc} \| & & \cap & & \\ \mathrm{id}\langle -2\rangle & \xrightarrow{\alpha} & \varepsilon_j \mathcal{L}Z_j \langle -1\rangle & & \rightsquigarrow \text{Cone}\,\alpha \\ \varepsilon_j \mathcal{L}Z_j[-1]\langle 1\rangle & \xrightarrow{\beta} & \mathrm{id}[1] & & \rightsquigarrow \text{Cone}\,\beta \\ \cap & & \| & & \end{array}$$

 $= \text{cone}\ \alpha$  $= \text{cone}\ \beta$

To each (r,s) tangle diagram D, we have a functor

$$\tilde{\varphi}(D): D^b(\oplus \mathcal{O}_i(\mathfrak{gl}_r)) \longrightarrow D^b(\oplus \mathcal{O}_i(\mathfrak{gl}_s))$$

__Thm__ (Stroppel)

T: oriented (r,s) tangle

D_1, D_2 : two of its diagram

$$\Longrightarrow \quad \tilde{\varphi}(D_1)\langle 3\gamma(D_1)\rangle = \tilde{\varphi}(D_2)\langle 3\gamma(D_2)\rangle$$

$$[\tilde{\varphi}(T)] = \varphi(T) : V_1^{\otimes r} \longrightarrow V_1^{\otimes s}$$

Let $\mathcal{H}(\mathfrak{gl}_n)$ be the Harish-Chandra category of $(U\mathfrak{gl}_n, U\mathfrak{gl}_n)$ bimodules

1) finitely generated
2) objects have finite length
3) " are locally finite w.r.t. the adjoint of $\mathfrak{gl}_n$

__Let__ ${}_{\lambda}^{(\infty)}\mathcal{H}^{1}_{\mu}(\mathfrak{gl}_n)$ be the subcat of $\mathcal{H}(\mathfrak{gl}_n)$

left action of $\mathfrak{gl}_n$ has generalised central character corresponding to int. dom. wt λ

right action has central character to μ.

$\vec{d} = (d_1, \dots, d_r)$ $\quad |\vec{d}| = n$

${}_i\mathcal{H}^1_{\vec{d}} = {}_\lambda\mathcal{H}^1_\mu$ where the stab. of $\lambda = S_i \times S_{n-i}$

" of $\mu = S_{d_1} \times \cdots \times S_{d_r}$

Th (FKS) $\mathbb{C}(q)\otimes_{\mathbb{Z}[q,q^{-1}]}\left[\bigoplus_{i=0}^{n} {}_i\mathcal{H}^1_{\vec{d}}(\mathfrak{gl}_n)\right] \cong V_{d_1}\otimes\cdots\otimes V_{d_r}$

the functors $\mathcal{E},\mathcal{F},\mathcal{K},\mathcal{K}^{-1}$ satisfy the functorial $U_q(\mathfrak{sl}_2)$-rel.

Rem ${}_i\mathcal{H}_{(1\cdots1)} \cong \mathcal{O}_i$

Let ${}_\lambda\tilde{\mathcal{L}}_\mu : {}_\lambda\mathcal{H}^1_\mu(\mathfrak{gl}_n) \longrightarrow \mathcal{O}_\lambda(\mathfrak{gl}_n)$

$${}_\lambda\tilde{\mathcal{L}}_\mu M = M\otimes_{U\mathfrak{gl}_n} M(\mu)$$

dominant Verma module

$${}_\lambda\tilde{\Pi}_\mu : \mathcal{O}_\lambda(\mathfrak{gl}_n) \longrightarrow {}_\lambda\mathcal{H}^1_\mu(\mathfrak{gl}_n)$$

$$M \longmapsto \mathrm{Hom}_{\mathbb{C}}(M(\mu), M)^{\text{l.f.}}$$

- In the case that μ is regular (trivial stabilizer) these functors are inverse equivalence of categories.

- The image of ${}_\lambda\tilde{\mathcal{L}}_\mu$ in $\mathcal{O}_\lambda(\mathfrak{gl}_n)$ is some subcategory ${}^\mu\mathcal{O}_\lambda(\mathfrak{gl}_n)$

(This is the subcat. of modules with proj. presentations, where the projectives allowed depend on the data λ & μ.

$$\bigoplus_i {}_i\widetilde{L}_{\vec d} = \widetilde{L}_{\vec d} : \bigoplus_i {}_i\mathcal{H}_{\vec d}(\mathfrak{gl}_n) \longrightarrow \bigoplus_i \mathcal{O}_i(\mathfrak{gl}_n)$$

$$\bigoplus_i \widetilde{\Pi}_{\vec d} = \widetilde{\Pi}_{\vec d} : \bigoplus_i \mathcal{O}_i(\mathfrak{gl}_n) \longrightarrow \bigoplus_i {}_i\mathcal{H}_{\vec d}(\mathfrak{gl}_n)$$

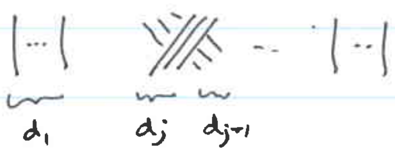
$\widetilde{L}_{\vec d}$ is not exact, its left derived functor

$$\mathcal{L}\widetilde{L} : D^<(\bigoplus_i {}_i\mathcal{H}_{\vec d}) \longrightarrow D^<(\bigoplus_i \mathcal{O}_i(\mathfrak{gl}_n))$$

Prop. 1) $[\mathcal{L}\widetilde{L}_{\vec d}] = L_{d_1}\otimes\cdots\otimes L_{d_r} : V_{d_1}\otimes\cdots\otimes V_{d_r} \to V_1^{\otimes n}$

2) $[\widetilde{\Pi}_{\vec d}] = \Pi_{d_1}\otimes\cdots\otimes\Pi_{d_r} : V_1^{\otimes n} \to V_{d_1}\otimes\cdots\otimes V_{d_r}$

Let $\widetilde{X}_{d_j,d_{j+1}}$ be the functor associated with



Prop. $D^<(\mathcal{O}^{(d_1,\dots,d_r)}_i(\mathfrak{gl}_n)) \xrightarrow{\widetilde{X}_{d_j,d_{j+1}}} D^<(\mathcal{O}^{(d_1,\dots,d_{j+1},d_j,\dots,d_r)}_i(\mathfrak{gl}_n))$

To an oriented $((d_1,\dots,d_r),(e_1,\dots,e_s))$ tangle diagram D

we define $\mathcal{E}_{col}(D) : \bigoplus_{i=0}^{|\vec d|} D^<({}_i\mathcal{H}_{\vec d}(\mathfrak{gl}(|\vec d|)))$

$$\to \bigoplus_{j=0}^{|\vec e|} D^<({}_i\mathcal{H}_{\vec e}(\mathfrak{gl}(|\vec e|)))$$

$$\mathcal{E}_{col}(D) = \widetilde{\Pi}_{\vec e}\circ\widetilde{\mathcal{E}}(\mathrm{cab}\,D)\circ\mathcal{L}\widetilde{L}_{\vec d}$$

Th T : framed oriented $(\vec{d}, \vec{e})$ tangle

Let D_1, D_2 be two of its diagram.

$$\Longrightarrow \widetilde{\mathcal{C}}_{col}(D_1)\langle 3\gamma(\mathrm{cab}\, D_1)\rangle \cong \widetilde{\mathcal{C}}_{col}(D_2)\langle 3\gamma(\mathrm{cab}\, D_2)\rangle$$